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Effect of ohmic dissipation on internal Alfvén–gravity waves in a conducting shear flow

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Internal Alfvén–gravity waves of small amplitude propagating in a Boussinesq, inviscid, adiabatic, finitely conducting fluid in the presence of a uniform transverse magnetic field in which the mean horizontal velocity $U(z)$ depends on height z only are considered. We find that the governing wave equation is singular only at the Doppler-shifted frequency $\Omega_d = 0$ and not at the magnetic singularities $\Omega_d = \pm\,\Omega_A$, where Ω_A is the Alfvén frequency. Hence the effect of ohmic dissipation is to prevent the resulting wave equation from having magnetic singularities. Asymptotic solutions of the wave equation, which is a fourth-order differential equation, are obtained. They show the presence of the magnetic Stokes points $\Omega_d = \pm\,\Omega_A$. The interpretation of upward and downward propagation of waves is also discussed.

To study the combined effect of electrical conductivity and the magnetic field on waves at the critical level, we have used the group-velocity approach and found that the waves are transmitted across the magnetic Stokes points but are completely absorbed at the hydrodynamic critical level $\Omega_d = 0$. The general expression for the momentum flux is mathematically complicated but will be simplified under the assumption

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2} \geqslant \frac{\partial^2 h}{\partial z^2},$$

where h is the perturbation magnetic field. In this approximation we find that the momentum flux is not conserved and the waves are completely absorbed at $\Omega_d = 0$.

The general theory is applied to a particular problem of flow over a sinusoidal corrugation and asymptotic solutions are obtained by applying the Laplace transformation and using the method of steepest descent.

1. Introduction

Much attention has been given in recent years to the detection and measurement of irregular motions in the D, E and lower F regions of the atmosphere and to the occurrence of irregular density distributions at the same heights. Many of these irregularities may have their origin in disturbances in the atmosphere, namely, in propagating atmospheric waves controlled by gravitational, compressive and Lorentz forces. The physical basis for such types of waves in the ionosphere has been presented by Lighthill (1960). Thus, internal wave phenomena in stratified or non-stratified flow with or without rotation in various situations have received considerable attention from a number of authors (see Bretherton 1966; Booker & Bretherton 1967). The governing mechanism of such waves in fluids is often attributed to the density stratification, rotation, temperature distribution, etc., and is of great practical importance in oceanography and meteorology (see Hines 1963).

The problem of the hydrodynamic stability of stratified shear flows, neglecting viscosity, although having antecedents in the work of Kelvin and Rayleigh, dates from G. I. Taylor's Adam's prize essay of 1915, which was published concurrently with a closely related investigation by Goldstein (1931). These two papers, dealing primarily with specific flow configurations, were followed closely by Synge's (1933) study of the general boundary-value problem and derived the equation for a small disturbance in the form (see Booker & Bretherton 1967)

$$w'' + \left(\frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2\right) w = 0, \tag{1.1}$$

where N is the Brunt–Väisälä frequency, U is the basic velocity and depends only on the vertical co-ordinate z, w is the vertical component of velocity, which is normal to the undisturbed flow, primes denote differentiation with respect to z, $c = \omega/k$ is the wave velocity, ω is the oscillation frequency and k is the wave-number. The stability of the flow on the basis of (1.1) was also discussed by Miles (1961) and Howard (1961) and they have shown that if $U'' \neq 0$ a sufficient condition for stability is that the Richardson number J_H be greater than one-quarter. Equation (1.1) has a singularity whenever $U-c = 0$ and this singularity is worse than that in the absence of gravity. The physical significance of (1.1) at the singular point where the velocity component of the small disturbance parallel to the undisturbed flow becomes infinite has been discussed by Booker & Bretherton (1967). However, in any real fluid, in the absence of gravity, viscosity will intervene to prevent the velocity component becoming infinite at the singular plane. Hence, it is necessary to invoke viscosity in order to get a physically meaningful small disturbance equation. Although this equation is not singular at $U-c = 0$ this point still has some significance in connexion with the asymptotic solutions because it is a Stokes point for such solutions. Koppel (1964) has shown that, in the case of density stratification which is mainly due to a temperature gradient with a non-zero thermal conductivity in the presence of gravity, it is necessary to include not only viscosity but also heat conductivity to prevent the resulting differential equation from having a singularity, and he gives an analytical asymptotic solution to the resulting sixth-order differential equation. The same problem has also been investigated numerically by Hazel (1967), who has shown that two of his solutions tend to the inviscid solutions asymptotically well away from the critical levels, but the other four viscous solutions are not negligible near this level. Hazel's (1967) analysis predicts a phase change in the disturbance across the layer as well as an attenuation if $J_H > \frac{1}{4}$. The effect of viscosity on gravity waves is also discussed by Yanowitch (1967) with reference to the atmosphere. He has found that there is a region in which the solution behaves like certain solutions of the inviscid problem. Yanowitch (1967) also predicts that the viscosity, in addition to damping the motion, causes reflexion of waves. All the above analyses pertain to the linear theory. However, recently Kelly & Maslowe (1970) have made a nonlinear critical-layer analysis. They have shown that, in contrast to the linear viscous analysis, no phase change occurs across the critical layer.

The importance of Alfvén waves in the upper atmosphere, particularly in the ionosphere, has been discussed by Lighthill (1960), who has given the physical significance of such waves. Recently, Rudraiah & Venkatachalappa (1972*b*, *c*, hereafter called RV *b*, *c*) have discussed Alfvén–gravity waves in a non-dissipative stratified shear flow and obtained the following differential equation for a small disturbance:

$$\frac{d^2w}{dz^2}+\frac{2k\Omega_A^2U_z}{\Omega_d(\Omega_d^2-\Omega_A^2)}\frac{dw}{dz}+\left[\frac{k^2N^2}{\Omega_d^2-\Omega_A^2}-\frac{kU_{zz}}{\Omega_d}-k^2-\frac{2k^2\Omega_A^2U_z^2}{\Omega_d^2(\Omega_d^2-\Omega_A^2)}\right]w=0, \quad (1.2)$$

where $\Omega_d = kU-\omega$ is the Doppler-shifted frequency, $\Omega_A = kA$ is the Alfvén frequency and A is the Alfvén velocity. Equation (1.2) is singular at $\Omega_d = 0, \pm\Omega_A$. That is, there are two magnetic singularities in addition to one hydrodynamic singularity. The corresponding problem in the presence of uniform rotation has also been discussed by Rudraiah & Venkatachalappa (1972*a*, hereafter RV *a*), who have shown that the corresponding wave equation is singular at

$$\Omega_d = 0,\ \pm\Omega_A,\ \pm\Omega\pm(\Omega^2+\Omega_A^2)^{\frac{1}{2}},$$

where Ω is the Coriolis frequency. These singularities imply that the horizontal velocity and magnetic field components of the small disturbance become infinite at the singular planes. In any real fluid the effects of viscous, thermal and ohmic dissipation may intervene to prevent this from happening. Thus it is interesting to see whether the presence of ohmic dissipation will remove the magnetic singularities of (1.2) or not and hence to get physically meaningful solutions for the horizontal velocity and the magnetic field components. The stability of this system is investigated by Rudraiah (1964).

Therefore, the aim of the present paper is to study the propagation of internal Alfvén–gravity waves in an inviscid stratified shear flow of a finitely conducting fluid in the absence of thermal diffusion but in the presence of a uniform magnetic field. The purpose of this study is to demonstrate that the singularity $\Omega_d = 0$ is purely due to the neglect of the effects of viscous and thermal diffusion and that the magnetic viscosity (i.e. ohmic dissipation) plays no part near the hydrodynamic critical layer. It is found that the governing wave equation is of order four and is not singular at the magnetic critical layers, but is singular at the hydrodynamic critical layer for neutrally stable disturbances. Thus the magnetic singularities of (1.2) are due to the neglect of ohmic dissipation. Solutions of the wave equation are obtained near the hydrodynamic critical layer. Two solutions are the same as the inviscid hydrodynamic solutions obtained by Booker & Bretherton (1967). Of the other two solutions, one tends to zero and the other behaves logarithmically at this level. From the hydrodynamic solutions we find that the wave amplitude is attenuated by a factor $\exp(-\mu_0 \pi)$, where $\mu_0 = (J_H - \frac{1}{4})^{\frac{1}{2}} > 0$ and J_H is the hydrodynamic Richardson number. In other words, the behaviour of waves near $\Omega_d = 0$ is independent of the electrical conductivity and magnetic field. Away from the critical level there exist waves which are similar to the waves in a perfectly conducting fluid discussed in RV c. Both near and away from the critical level there are two waves similar to those in a perfectly conducting fluid. Also, there exist two other waves which cannot be neglected near the level $\Omega_d = \Omega_A$ but tend to zero as z becomes large.

In the present paper we also discuss the effect of electrical conductivity on the mechanism of absorption, reflexion and transmission of waves. This is done by examining the motion of the wave packets near the critical levels. It is shown that as a wave propagates vertically through the hydrodynamic critical level it is strongly attenuated.

However, the energy flux near this critical level becomes infinite. This infinite energy may be due to the neglect of viscosity and heat conduction and could be removed by including these dissipative effects. Work is in progress to include these dissipative effects.

The solution of the wave equation near the magnetic critical layers is obtained following the analysis of Koppel (1964). It becomes difficult to find, in general, an expression for the total Reynolds stress or the momentum flux. However, we note that it is possible to express the Reynolds stress in terms of w, the vertical perturbation velocity, if the perturbation magnetic field varies much less in the vertical direction than in the two horizontal directions. This implies that

$$\frac{\partial^2 h}{\partial z^2} \ll \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2},$$

where h is the perturbation magnetic field. In this approximation we also find, from the group-velocity approach, that the waves are absorbed at the hydrodynamic critical level. The total momentum flux is not conserved in contrast to the perfectly conducting fluid discussed in RV c. The non-conservation of momentum flux is due to the dissipative effect of magnetic viscosity (see Chandrasekhar 1961, §39; Hughes & Young 1966, §13).

Finally, we consider formally a specific example, the time development of a stationary train of waves above a corrugation in the lower boundary which is introduced suddenly at time $t = 0$. The flow field for this problem a long time later is also discussed.

The results of the present problem are of geophysical and astrophysical interest. One geophysical application is concerned with the propagation of internal Alfvén–gravity waves from the troposphere to the ionosphere. The results of the present paper are obtained on the assumption that the electrical conductivity is homogeneous and isotropic, whereas the ionospheric conductivity in general is not isotropic (Hines 1963). To apply the results of this paper to the geophysical problem we discuss the validity of the assumption of homogeneity and isotropy. In the ionosphere, in addition to viscous and heat dissipation, there will be magnetic dissipation due to the magnetic diffusivity, and the magnetic energy dissipation per unit volume is given by $\mathbf{J}.\mathbf{E}'$, where $\mathbf{J}$ is the current density, $\mathbf{E}' = \mathbf{E} + \mathbf{q} \times \mathbf{B}$, $\mathbf{E}$ is the electric field, $\mathbf{q} \times \mathbf{B}$ is the induced electric field, called the dynamo field, $\mathbf{q}$ is the velocity of conducting fluid induced by the internal Alfvén–gravity waves and $\mathbf{B}$ is the geomagnetic induction. The current density has a component $J_{\parallel} = \sigma_0 E_{\parallel}$ directed parallel to $\mathbf{B}$, where σ_0 is the longitudinal electrical conductivity and $E_{\parallel}$ is the component of $\mathbf{E}$ parallel to $\mathbf{B}$. It also has a component $J_{\perp} = \sigma_1 E_{\perp}$ transverse to $\mathbf{B}$ in the direction of the transverse component of $\mathbf{E}'$, where σ_1 is the Pedersen conductivity (see Hines 1963). We note that the current density will in addition have a Hall component transverse to both $\mathbf{B}$ and $\mathbf{E}'$ but that component is relatively small in the ionosphere and in any event contributes nothing to $\mathbf{J}.\mathbf{E}'$. Since the current density is nearly solenoidal, we have

$$|J_{\parallel}/L_{\parallel}| \approx |J_{\perp}/L_{\perp}|,$$

where $L_{\parallel}$ and $L_{\perp}$ are respectively the characteristic lengths of $J_{\parallel}$ along $\mathbf{B}$ and of $J_{\perp}$ across $\mathbf{B}$, and from this it follows that

$$|E_{\parallel}| \approx |E'_{\perp}|\,\sigma_1 L_{\parallel}/\sigma_0 L_{\perp}.$$

It follows in turn that

$$J_{\parallel}E_{\parallel} \approx J_{\perp}E'_{\perp}(\sigma_1/\sigma_0)\,L_{\parallel}^2/L_{\perp}^2.$$

Since σ_1 is less than σ_0 by four orders of magnitude or more, it follows that the energy dissipation derives primarily from the $J_{\perp}E'_{\perp}$ contribution to $\mathbf{J}.\mathbf{E}'$ unless $L_{\perp} \ll L_{\parallel}$. In our physical model $E_{\perp}$ results almost entirely from the $\mathbf{q} \times \mathbf{B}$ contribution to it and so the energy dissipation rate per unit volume is given approximately by $\sigma_1 q_{\perp}^2 B^2$, where $q_{\perp}$ is the component of $\mathbf{q}$ perpendicular to $\mathbf{B}$. In the present paper $q_{\perp} \equiv U$, which is in the x direction, and the magnetic induction $\mathbf{B}$ is in the y direction. Since the main contribution to the current density comes from $\mathbf{q} \times \mathbf{B}$ we can neglect the $J_{\parallel}$ component and consider only the $J_{\perp}$ $(= \sigma_1 UB)$ component and assume that $\sigma_1 \equiv \sigma$ is isotropic. In this approximation the results of the present paper are applicable to the geophysical problem discussed above.

2. Derivation of wave equation

The physical model consists of an incompressible, inviscid, heterogeneous, finitely conducting fluid occupying the region $Oxyz$ such that the axis Oz is vertical. Let the fluid have a mean density with vertical structure

$$d(\log \rho_0)/dz = -\beta. \tag{2.1}$$

To derive the wave equation for the motion of a finitely conducting fluid in the presence of a transverse magnetic field with the above vertical density stratification the following assumptions are made.

(*a*) The motion is three-dimensional. The fluid is inviscid, finitely conducting and adiabatic.

(*b*) The Boussinesq approximation.

(*c*) The perturbation velocities (u, v, w) from the basic $U(z)$ state in the x direction and the perturbation magnetic field (h_x, h_y, h_z) from the basic uniform applied magnetic field H_0 in the y direction, which is transverse to the mean flow, are so small that

$$\left|u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right|,$$

$$\left|h_x\frac{\partial}{\partial x}+h_y\frac{\partial}{\partial y}+h_z\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+H_0\frac{\partial}{\partial y}\right|.$$

We note that the problem of the propagation of internal Alfvén–gravity waves in a conducting fluid with an aligned magnetic field is similar to the case of a transverse magnetic field (i.e. in the y direction) except that Ω_A, which equals Al in the case of a transverse magnetic field, is replaced by Ak in the case of an aligned magnetic field, A being the Alfvén velocity and k and l the wavenumbers in the x and y directions respectively. We further note that, as in the case of a perfectly conducting fluid (RVc), if the applied magnetic field H_0 is in the z direction the basic horizontal velocity $U(z)$ has to be uniform to satisfy the magnetic induction equation. Since the aim of the present analysis is to consider shear flow (i.e. $dU/dz \neq 0$) we avoid an applied magnetic field in the z direction. Under these assumptions the linearized equations of motion are

$$\begin{bmatrix} \rho_0 D & 0 & \rho_0(D_3 U) & D_1 & 0 & -\mu H_0 D_2 & 0 & 0 \\ 0 & \rho_1 D & 0 & D_2 & 0 & 0 & -\mu H_0 D_2 & 0 \\ 0 & 0 & \rho_0 D & D_3 & g & 0 & 0 & -\mu H_0 D_2 \\ D_1 & D_2 & D_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_3\rho_0 & 0 & D & 0 & 0 & 0 \\ -H_0 D_2 & 0 & 0 & 0 & 0 & D-\nu_m\nabla^2 & 0 & -D_3 U \\ 0 & -H_0 D_2 & 0 & 0 & 0 & 0 & D-\nu_m\nabla^2 & 0 \\ 0 & 0 & -H_0 D_2 & 0 & 0 & 0 & 0 & D-\nu_m\nabla^2 \\ 0 & 0 & 0 & 0 & 0 & D_1 & D_2 & D_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ P \\ \rho \\ h_x \\ h_y \\ h_z \end{bmatrix} = 0, \tag{2.2}$$

where P is the total perturbation pressure, ρ is the perturbation density, g is the acceleration due to gravity, $\nu_m = 1/\mu\eta_0$ is the magnetic diffusivity, μ is the magnetic permeability, η_0 is the electrical conductivity,

$$D = \frac{\partial}{\partial t}+UD_1, \quad D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial y}, \quad D_3 = \frac{\partial}{\partial z},$$

and
$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

By eliminating u, v, P, ρ, h_x, h_y and h_z from (2.2), we obtain a single wave equation

$$D^5(\nabla^2 w) - D^4\left[\nu_m\frac{\partial^4 w}{\partial z^4}+\nu_m\nabla_1^2\left(\nabla^2 w+\frac{\partial^2 w}{\partial z^2}\right)+\frac{d^2U}{dz^2}\frac{\partial w}{\partial x}\right]$$
$$+D^3\left[N^2\nabla_1^2 w+\nu_m\frac{d^2U}{dz^2}\frac{\partial}{\partial x}(\nabla^2 w)-A^2\frac{\partial^2}{\partial y^2}(\nabla^2 w)+2\nu_m\frac{d^3U}{dz^3}\frac{\partial^2 w}{\partial x\,\partial z}-\nu_m\frac{d^4U}{dz^4}\frac{\partial w}{\partial x}\right]$$

$$-D^2\left[\nu_m N^2\nabla_1^2(\nabla^2 w)+2\nu_m\frac{dU}{dz}\frac{\partial}{\partial z}\left(\frac{d^2U}{dz^2}\frac{\partial^2 w}{\partial x^2}\right)+\nu_m\left(\frac{d^2U}{dz^2}\right)^2\frac{\partial^2 w}{\partial x^2}\right.$$
$$\left.-A^2\frac{\partial^3}{\partial x\,\partial y^2}\left(2\frac{dU}{dz}\frac{\partial w}{\partial z}+\frac{d^2U}{dz^2}w\right)\right]+D\left[2\nu_m\left(\frac{dU}{dz}\right)^2\frac{d^2U}{dz^2}\frac{\partial^3 w}{\partial x^3}\right.$$
$$\left.+2\nu_m\frac{\partial}{\partial x}\nabla_1^2\left(2\frac{dU}{dz}\frac{\partial w}{\partial z}+\frac{d^2U}{dz^2}w\right)-2A^2\left(\frac{dU}{dz}\right)^2\frac{\partial^4 w}{\partial x^2\partial y^2}\right]$$
$$-6\nu_m\left(\frac{dU}{dz}\right)^2 N^2\nabla_1^2\left(\frac{\partial^2 w}{\partial x^2}\right)=0, \qquad (2.3)$$

where $A = H_0(\mu/\rho_0)^{\frac{1}{2}}$ is the Alfvén velocity and

$$\nabla_1^2 = \partial^2/\partial x^2+\partial^2/\partial y^2.$$

The hydromagnetic wave equation (2.3) for a finitely conducting fluid is of order eight, whereas the corresponding equation for a perfectly conducting fluid discussed in RV *c* is of order six, which is of importance in the determination of critical levels (see § 3 below).

We can assume a sinusoidal wave of small amplitude of the form

$$w(x,y,z,t) = \hat{w}(z)\exp\left[i(kx+ly-\omega t)\right], \qquad (2.4)$$

where ω is the wave frequency and k and l are the wavenumbers in the x and y directions respectively. Then $\hat{w}(z)$ satisfies the equation

$$i\nu_m\frac{d^4\hat{w}}{dz^4}+\left[\frac{\Omega_d^2-\Omega_A^2}{\Omega_d}-\frac{i\nu_m(k^2+l^2)}{\Omega_d^2}(2\Omega_d^2-N^2)-\frac{ik\nu_m}{\Omega_d}\frac{d^2U}{dz^2}\right]\frac{d^2\hat{w}}{dz^2}$$
$$+\left[-\frac{2ik\nu_m}{\Omega_d}\frac{d^3U}{dz^3}+\frac{2ik^2\nu_m}{\Omega_d^2}\frac{dU}{dz}\frac{d^2U}{dz^2}+\frac{2\{\Omega_A^2\,\Omega_d-2i\nu_m(k^2+l^2)N^2\}}{\Omega_d^3}k\frac{dU}{dz}\right]\frac{d\hat{w}}{dz}$$
$$+\left[\frac{N^2-\Omega_d^2}{\Omega_d^2}\{\Omega_d-i\nu_m(k^2+l^2)\}(k^2+l^2)+\frac{\Omega_A^2}{\Omega_d}(k^2+l^2)-\{2\Omega_A^2\,\Omega_d\right.$$
$$-6iN^2(k^2+l^2)\,\nu_m\}\left(\frac{k\,dU/dz}{\Omega_d^2}\right)^2+\frac{1}{\Omega_d^3}\left\{-\Omega_d^3+i\nu_m(k^2+l^2)\,\Omega_d^2\right.$$
$$\left.+\left(\Omega_A^2+ik\frac{d^2U}{dz^2}\nu_m\right)\Omega_d-2i\nu_m\left(k\frac{dU}{dz}\right)^2-2iN^2(k^2+l^2)\,\nu_m\right\}k\frac{d^2U}{dz^2}$$
$$\left.+\frac{2ik^2\nu_m}{\Omega_d^2}\frac{dU}{dz}\frac{d^3U}{dz^3}-\frac{ik\nu_m}{\Omega_d}\frac{d^4U}{dz^4}\right]\hat{w}=0, \qquad (2.5)$$

where $\Omega_A = Al$.

We note that this fourth-order wave equation (2.5) tends to a second-order wave equation ,which is similar to (1.2), in the limit $\nu_m\rightarrow 0$. Another important feature is that in the case of a perfectly conducting fluid discussed in RV *c* the wave equation is singular at both the hydrodynamic critical layer $\Omega_d = 0$ and the magnetic critical layers $\Omega_d = \pm\,\Omega_A$, whereas in the case of a finitely conducting fluid discussed here the wave equation (2.5) is singular only at $\Omega_d = 0$ and not at $\Omega_d = \pm\,\Omega_A$. The removal of the singularities can be attributed to the inclusion of higher order derivatives representing dissipative effects. Therefore the effect of the ohmic dissipation is to remove the magnetic singularities. The magnetic diffusivity does not play any part in removing the hydrodynamic singularity. This is analogous to the part played by viscosity in the case of hydrodynamic flow discussed by Koppel (1964) and Hazel (1967). In that case Koppel (1964) has shown that to remove the hydrodynamic singularity in addition to viscosity we should also consider the thermal diffusivity. The same situation may also prevail even in the case of a stratified conducting fluid in the presence of gravity, i.e. inclusion of the thermal diffusivity and viscosity in addition to the magnetic viscosity may also remove the hydrodynamic singularity of (2.5). Work relating to this is in progress.

3. Solution of wave equation

The singularity of (2.5) at the critical level $\Omega_d = 0$ may evidently be regarded as a consequence of the loss of higher order derivatives owing to the neglect of viscous and thermal dissipative effects and can presumably be resolved by a boundary-layer-type analysis as discussed by Hazel (1967). This singularity, as Miles (1961) has pointed out in connexion with the critical level for internal gravity waves in a shear flow (see Acheson 1972), may also be regarded as a consequence of restricting our attention to a single sinusoidal component given by (2.4). Accordingly, by posing an initial-value problem and then determining its asymptotic solution as $t \to \infty$ (see RV*c*) we should be able to match the solutions on the two sides of the critical level even in the absence of viscous and thermal dissipation. It has, however, proved possible to resolve the singularity by simpler means following Booker & Bretherton (1967). The method involves allowing the phase velocity c to have a small imaginary part $c_i > 0$ so that the amplitude of the wave at any station is slowly growing with time. By thus investigating the solution of (2.5) near $z = z_0$ (the level where $\Omega_d = 0$) and then taking the limit $c_i \to 0$ we obtain a matching condition connecting the solutions on the two sides of the critical level. The detailed matching process is omitted here for it is similar to that of RV*c*. Asymptotic solutions near the hydromagnetic critical layers $\Omega_d = \pm \Omega_A$ are also obtained. The physical significance of these solutions will be explained in the next section.

The solution of (2.5) near $\Omega_d = 0$, obtained by the method of Frobenius, is

$$\hat{w} = A_1\psi_1 + B_1\psi_2 + C_1\psi_3 + D_1\psi_4, \tag{3.1}$$

where

$$\left.\begin{aligned} \psi_1 &= (z-z_0)^{\frac{1}{2}+i\mu_0}[1+a_1(z-z_0)+\ldots], \\ \psi_2 &= (z-z_0)^{\frac{1}{2}-i\mu_0}[1+b_1(z-z_0)+\ldots], \\ \psi_3 &= (z-z_0)^3[1+c_1(z-z_0)+\ldots], \\ \psi_4 &= \psi_3 \log(z-z_0) + (z-z_0)^2[1+d_1(z-z_0)+\ldots], \end{aligned}\right\} \tag{3.2}$$

a_1, b_1, c_1 and d_1 are known constants such that

$$a_1, b_1, c_1, d_1 = \frac{-i(r-1)\,\Omega_A^2}{2\nu_m k(dU/dz)\,[2r(r-1)+J_H]} \quad \text{with} \quad r = \tfrac{1}{2}+i\mu_0, \;\; \tfrac{1}{2}-i\mu_0, \; 3, \; 2,$$

$$\mu_0 = (J_H - \tfrac{1}{4})^{\frac{1}{2}},$$

$$J_H = \frac{N^2}{(dU/dz)^2}\left(1+\frac{l^2}{k^2}\right)$$

is the hydrodynamic Richardson number.

We note that the solutions ψ_1 and ψ_2 are similar to inviscid hydrodynamic solutions given by Booker & Bretherton (1967). Hence, the first solution represents an upward-travelling wave and the second solution represents a downward-travelling wave. This situation was also observed by Hazel (1967) in the discussion of hydrodynamic internal gravity waves in the presence of heat conduction and viscosity. He showed that two of his solutions away from the critical level are similar to those given by Booker & Bretherton (1967) near the critical level. We also note that the remaining two solutions ψ_3 and ψ_4 in (3.1) tend to zero as $z \to z_0$.

Now, if we fix the branch of $(z-z_0)^{\frac{1}{2}+i\mu_0}$ by choosing

$$(z-z_0)^{\frac{1}{2}\pm i\mu_0} = |z-z_0|^{\frac{1}{2}} \exp\left(\pm i\mu_0 \log|z-z_0|\right) \quad \text{for} \quad z > z_0, \tag{3.3}$$

it then follows that

$$(z-z_0)^{\frac{1}{2}\pm i\mu_0} = -i\exp(\mu_0\pi)\,|z-z_0|^{\frac{1}{2}} \exp\left(\mp i\mu_0 \log|z-z_0|\right) \quad \text{for} \quad z < z_0. \tag{3.4}$$

Thus the magnitude of the first two terms in (3.1) is not the same at a given distance above and below the critical layer $\Omega_d = 0$ but differs by a factor of $\exp(-\mu_0\pi)$. In other words, they are attenuated on passage through the critical layer. The magnitude of the third term in (3.1) remains the same and hence is not attenuated on passage through the critical layer. Whereas the square of the fourth term in (3.1) differs by an amount $D_1^2\pi^2$ and hence is attenuated there.

The above solutions are correct near the hydrodynamic critical layer $\Omega_d = 0$. In the remaining part of this section we try to find, following Koppel (1964), the solution of (2.5) near the magnetic critical layers $\Omega_d = \pm\Omega_A$. For this, the trial function

$$\hat{w}(z) = A_0(z)\exp\left[(\alpha/i\nu_m)^{\frac{1}{2}} B_0(z)\right] \tag{3.5}$$

with $\alpha = (k^2+l^2)^{\frac{1}{2}}$ is substituted into the differential equation.

Substituting (3.5) into (2.5) and equating to zero the terms of order $\alpha/i\nu_m$ we get

$$\alpha\Omega_d^2 A_0 B_0'^4 + \Omega_d(\Omega_d^2-\Omega_A^2)B_0'^2 A_0 = 0, \tag{3.6}$$

where a prime denotes the differentiation with respect to z. If we suppose that $\Omega_d \neq 0$ and $A_0 \neq 0$ this becomes

$$\alpha\Omega_d B_0'^4 + (\Omega_d^2-\Omega_A^2)B_0'^2 = 0.$$

The solutions of this equation are

$$B_0'^2 = 0, \quad B_0'^2 = -(\Omega_d^2-\Omega_A^2)/\alpha\Omega_d.$$

Now equating to zero the terms of order $(\alpha/i\nu_m)^{\frac{1}{2}}$, we obtain

$$\alpha\Omega_d^2(4A_0'B_0'^3 + 6A_0B_0'^2B_0'') + \Omega_d(\Omega_d^2-\Omega_A^2)(2A_0'B_0' + A_0B_0'') + 2\Omega_A^2 k(dU/dz)A_0B_0' = 0.$$

If $B_0'^2 = 0$, this equation is identically satisfied. The case

$$B_0'^2 = -(\Omega_d^2-\Omega_A^2)/\alpha\Omega_d$$

leads to the following equation:

$$\frac{2A_0'}{A_0} = -\frac{5B_0''}{B_0'} + \frac{2\Omega_A^2 k\, dU/dz}{\Omega_d(\Omega_d^2-\Omega_A^2)}.$$

Integrating this, we get

$$A_0 \propto \Omega_d^{\frac{1}{4}}/(\Omega_d^2-\Omega_A^2)^{\frac{3}{4}}.$$

Hence we obtain two solutions

$$\left.\begin{aligned}\phi_3 &= \Omega_d^{\frac{1}{4}}(\Omega_d^2-\Omega_A^2)^{-\frac{3}{4}}\exp\left[-\left(\frac{i\alpha}{\nu_m}\right)^{\frac{1}{2}}\int\left(\frac{\Omega_d^2-\Omega_A^2}{\alpha\Omega_d}\right)^{\frac{1}{2}}dz\right],\\ \phi_4 &= \Omega_d^{\frac{1}{4}}(\Omega_d^2-\Omega_A^2)^{-\frac{3}{4}}\exp\left[\left(\frac{i\alpha}{\nu_m}\right)^{\frac{1}{2}}\int\left(\frac{\Omega_d^2-\Omega_A^2}{\alpha\Omega_d}\right)^{\frac{1}{2}}dz\right].\end{aligned}\right\} \tag{3.7}$$

Since for two cases the equation determined by letting the terms of order $(\alpha/i\nu_m)^{\frac{1}{2}}$ vanish is identically satisfied, we must go to terms $O(1)$ in (2.5) for these cases. When $B_0'^2=0$, this gives

$$\Omega_d(\Omega_d^2-\Omega_A^2)A_0''+2k\frac{dU}{dz}\Omega_A^2A_0'+\left[(N^2+\Omega_A^2-\Omega_d^2)\,\Omega_d\alpha^2-\frac{2\Omega_A^2(k\,dU/dz)^2}{\Omega_d}\right]A_0=0. \tag{3.8}$$

This is the same as the small disturbance equation obtained in RV*c* for a perfectly conducting inviscid fluid. In other words, the magnetic viscosity has no effect near the critical level $\Omega_d=0$ and away from $\Omega_d=\Omega_A$ and is predominant only at $\Omega_d=\Omega_A$. The two solutions determined from (3.8) will be called ϕ_1 and ϕ_2. Therefore, the solutions of (3.8) near the critical level $\Omega_d=\Omega_A$, following RV*c*, are

$$\left.\begin{aligned}\phi_1 &= 1+c_1(z-z_1)+\dots,\\ \phi_2 &= \phi_1\log(z-z_1)+\sum_{k=0}^{\infty}\left(\frac{\partial c_k}{\partial r}\right)_{r=0}(z-z_1)^k,\end{aligned}\right\} \tag{3.9}$$

where z_1 is the position of the critical layer $\Omega_d=\Omega_A$. From the solution ϕ_2 it follows that the amplitude of the wave at a given distance from $\Omega_d=\Omega_A$ on either side is not the same and the square of the amplitude differs by a factor π^2.

Similar solutions can be obtained near the lower magnetic critical layer at z_2 corresponding to $\Omega_d=-\Omega_A$. We note that the asymptotic solutions (3.7) are singular at $\Omega_d=\pm\Omega_A$, even though the governing differential equation (2.5) is not singular at such points. These points are called magnetic Stokes points. A good representation of the solutions near the Stokes points is lengthy and will be presented in a subsequent paper.

The above solutions are correct in the case of shear flow. When the basic flow is uniform, however, the solution of (2.5) is

$$\hat{w}=A_2e^{im_1z}+B_2e^{-im_1z}+C_2e^{im_2z}+D_2e^{-im_2z}, \tag{3.10}$$

where

$$m_1,m_2=\frac{1}{(2\nu_m)^{\frac{1}{2}}}\left[\frac{\nu_m\alpha^2(N^2-2\Omega_d^2)}{\Omega_d^2}-\frac{i(\Omega_d^2-\Omega_A^2)}{\Omega_d}\right.$$
$$\left.\pm\left\{4\Omega_A^2+\left(\frac{\nu_m\alpha^2N^2}{\Omega_d^2}+\frac{i(\Omega_d^2+\Omega_A^2)}{\Omega_d}\right)^2\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

The branches for m_1 and m_2 can be chosen by requiring that, if $c_i > 0$, $m_{1i} > 0$ and $m_{2i} > 0$, where c_i, m_{1i} and m_{2i} are the imaginary parts of c, m_1 and m_2 respectively, c being the phase velocity.

4. Energy flow and group velocity near the critical level

In this section we discuss the phenomenon of absorption of waves near the critical levels using the concepts of energy flow and group velocity.

4.1. *Energy flow*

It is found that it is not possible to express the momentum flux in terms of $\hat{w}$ and hence momentum transfer to the mean flow cannot be discussed analytically, but can be determined using numerical analysis. In this section, however, instead of finding the momentum flux, we find the energy flow near the critical level $\Omega_d = 0$. It is found that

$$P = -\frac{\rho_0}{\alpha^2}\Bigg[-\nu_m\frac{d^3\hat{w}}{dz^3}+\left\{i\Omega_d+\frac{k\nu_m(d^2U/dz^2)}{\Omega_d}+\alpha^2\nu_m-\frac{\alpha^2\nu_m N^2}{\Omega_d^2}-\frac{i\Omega_A^2}{\Omega_d}\right\}\frac{d\hat{w}}{dz}$$
$$+\left\{-ik\frac{dU}{dz}+\frac{k\nu_m}{\Omega_d}\frac{d^3U}{dz^3}-\frac{k^2\nu_m}{\Omega_d^2}\frac{dU}{dz}\frac{d^2U}{dz^2}+\frac{2k\alpha^2\nu_m N^2}{\Omega_d^3}\left(\frac{dU}{dz}\right)+\frac{i\Omega_A^2}{\Omega_d^2}\frac{k\,dU}{dz}\right\}\hat{w}\Bigg],$$

where P is the perturbation pressure.

Using (3.2) we obtain the energy flow $\overline{Pw}$, which for the first two solutions becomes infinite when $\Omega_d \to 0$, where the overbar represents the average over a horizontal wavelength. Thus when dU/dz = constant, we have for the first solution of (3.2)

$$\overline{Pw} = \begin{cases} -\dfrac{\rho_0|A_1|^2}{2\alpha^2}\left[\dfrac{\alpha^2\nu_m}{2}-\dfrac{\mu_0(\Omega_d^2-\Omega_A^2)}{k(dU/dz)\,|z-z_0|}\right] & \text{for} \quad z > z_0, \\ -\dfrac{\rho_0|A_1|^2 e^{2\mu_0\pi}}{2\alpha^2}\left[\dfrac{\alpha^2\nu_m}{2}-\dfrac{\mu_0(\Omega_d^2-\Omega_A^2)}{k(dU/dz)\,|z-z_0|}-\dfrac{4\nu_m\alpha^2N^2}{(k\,dU/dz)^2\,|z-z_0|^2}\right] & \text{for} \quad z < z_0. \end{cases}$$

For the second solution

$$\overline{Pw} = \begin{cases} -\dfrac{\rho_0|B_1|^2}{2\alpha^2}\left[\dfrac{\alpha^2\nu_m}{2}+\dfrac{\mu_0(\Omega_d^2-\Omega_A^2)}{k(dU/dz)\,|z-z_0|}\right] & \text{for} \quad z > z_0, \\ -\dfrac{\rho_0|B_1|^2 e^{-2\mu_0\pi}}{2\alpha^2}\left[\dfrac{\alpha^2\nu_m}{2}+\dfrac{\mu_0(\Omega_d^2-\Omega_A^2)}{k(dU/dz)\,|z-z_0|}-\dfrac{4\nu_m\alpha^2N^2}{(k\,dU/dz)^2\,|z-z_0|^2}\right] & \text{for} \quad z < z_0. \end{cases}$$

In these two cases $\overline{Pw} \to \infty$ as $z \to z_0$. For the third solution of (3.2)

$$\overline{Pw} = -\frac{\rho_0|C_1|^2(z-z_0)^3}{2\alpha^2}\left[-6\nu_m+\alpha^2\nu_m\left\{3(z-z_0)^2-\frac{N^2}{(k\,dU/dz)^2}\right\}\right],$$

which tends to zero as $z \to z_0$. Similarly, for the fourth solution $\overline{Pw} \to 0$ as $z \to z_0$.

In the case of uniform basic flow $\overline{Pw}$ takes the form

$$\overline{Pw} = -\frac{\rho_0}{2\alpha^2}\mathrm{Re}\left[-\nu_m\frac{d^3\hat{w}}{dz^3}+\left\{i\Omega_d+\alpha^2\nu_m-\frac{\alpha^2\nu_m N^2}{\Omega_d^2}-i\frac{\Omega_A^2}{\Omega_d^2}\right\}\frac{d\hat{w}}{dz}\right]\hat{w}^*,$$

where $\hat{w}$ is given by (3.10). We note, following the analysis of RV*c*, that $\overline{Pw}$ is positive for the first and third solutions of (3.10) and hence the wave energy is flowing upwards. In other words, they represent upward-propagating waves. Similarly, the second and fourth solutions represent downward-propagating waves.

Further, we note that the magnetic diffusivity is not sufficient to remove the infinite energies, which could be removed by including viscosity and heat conduction.

4.2. *Group velocity*

The full dispersion relation satisfied by an internal Alfvén–gravity wave with horizontal wavenumbers k and l and vertical wavenumber m is

$$\begin{aligned}
&-i\Omega_d^5(\alpha^2+m^2)-\Omega_d^4[\nu_m(\alpha^2+m^2)^2+ikU'']\\
&\quad+i\Omega_d^3[N^2\alpha^2+ik\nu_m U''(\alpha^2+m^2)+\Omega_A^2(\alpha^2+m^2)+2\nu_m mkU'''+ik\nu_m U^{\mathrm{iv}}]\\
&\quad+\Omega_d^2[\alpha^2(\alpha^2+m^2)\nu_m N^2-2\nu_m U'(k^2U'''+iU''mk^2)\\
&\qquad\qquad-\nu_m k^2U''^2+ik\Omega_A^2(2imU'+U'')]\\
&\quad-i\Omega_d[2i\nu_m k^3U'^2U''+2i\nu_m kN^2\alpha^2(U''+2imU')+2A^2k^2l^2U'^2]\\
&\qquad\qquad-6\nu_m k^2\alpha^2U'^2N^2=0. \qquad (4.1)
\end{aligned}$$

In a slowly varying medium (i.e. if $U(z)$ and N do not vary very much over a wavelength), however, an internal Alfvén–gravity wave satisfies the dispersion relation

$$\frac{\Omega_d^3}{m^2+\alpha^2}-i\nu_m\Omega_d^2-\left[\frac{\Omega_A^2(m^2+\alpha^2)+N^2\alpha^2}{(m^2+\alpha^2)^2}\right]\Omega_d+\frac{i\nu_m\alpha^2N^2}{m^2+\alpha^2}=0. \qquad (4.2)$$

We note that this relation reduces to one obtained by RV*c* when $\nu_m\rightarrow 0$, i.e. in the perfectly conducting case, and to one obtained by Booker & Bretherton (1967) when $l\rightarrow 0$, $\nu_m\rightarrow 0$ and $\Omega_A\rightarrow 0$, i.e. in the hydrodynamic case. We obtain the same dispersion relation (4.2) for the case of uniform basic flow with Ω_d constant.

When $\Omega_d\rightarrow 0$, we observe that $m\rightarrow\infty$. Also when $\nu_m\neq 0$ and Ω_d is real, m is complex. Equation (4.2) is precisely the frequency relation for plane internal gravity waves in uniformly stratified finitely conducting media. We note that, as in the hydrodynamic case (see Bretherton 1966), this comparison is only locally valid.

From (4.2) we find that $\partial\omega/\partial m$ tends to zero as $\Omega_d\rightarrow 0$, whereas it is finite at $\Omega_d=\Omega_A$. Hence, waves are completely absorbed at the critical layer $\Omega_d=0$ without being reflected or transmitted and they are transmitted through the magnetic critical levels $\Omega_d=\pm\Omega_A$. Comparing these results with those for the perfectly conducting case discussed by RV*c*, where the waves are completely absorbed at the magnetic critical levels, we find that the effect of magnetic diffusivity is to prevent the total absorption at the magnetic critical levels.

5. Approximation to the momentum flux

In this section we find an expression for the momentum flux using the approximation

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2} \gg \frac{\partial^2 h}{\partial z^2}. \tag{5.1}$$

This approximation can be justified physically using a result obtained by Acheson & Hide (1973). By considering a plane sinusoidal wave of the form $\exp[i(kx+ly+mz+\omega t)]$ the above approximation leads to the fact that $m/(k^2+l^2)^{\frac{1}{2}} \ll 1$; Acheson & Hide (1973) have shown that the ratio $|m|/(k^2+l^2)^{\frac{1}{2}}$ associated with a slow hydromagnetic wave in the process of being captured at its critical level is small before effects due to ohmic dissipation become important. Hence the approximation (5.1) is valid in the case of slow hydromagnetic waves. Under this approximation, assuming sinusoidal variations of the form (2.4), the vertical disturbance velocity satisfies the equation

$$\Omega_d^2\frac{d^2\hat{w}}{dz^2}+\left[\alpha^2\left\{N^2+\frac{\Omega_d\Omega_A^2}{\Omega_d-i\nu_m\alpha^2}-\Omega_d^2\right\}-k\Omega_d\frac{d^2U}{dz^2}\right]\hat{w}=0, \tag{5.2}$$

where $\Omega_d = 0$ is the regular singular point of this equation. The solution of this equation near $\Omega_d = 0$ (i.e. $z = z_0$) is

$$\hat{w} = A_3(z-z_0)^{\frac{1}{2}+i\mu_0}\{1+a_{11}(z-z_0)+a_{12}(z-z_0)^2+\ldots\}$$
$$+B_3(z-z_0)^{\frac{1}{2}-i\mu_0}\{1+b_{11}(z-z_0)+b_{12}(z-z_0)^2+\ldots\}, \tag{5.3}$$

where $\quad a_{11}, a_{12} = \dfrac{1}{2r}\left[\dfrac{U_{zz}}{U_z}(1+J_H)+\dfrac{J_M kU_z}{i\nu_m\alpha^2}\right]\quad$ with $\quad r = \frac{1}{2}+i\mu_0,\ \frac{1}{2}-i\mu_0,$

$$J_M = \frac{\Omega_A^2}{(dU/dz)^2}\left(1+\frac{l^2}{k^2}\right).$$

Solution (5.3) is similar to the hydrodynamic solution of Booker & Bretherton and the behaviour of the waves will be the same as that of the hydrodynamic waves near the critical level $\Omega_d = 0$. In other words, under the approximation (5.1) the magnetic field has no effect on the waves near $\Omega_d = 0$.

Further, the solution of (5.2) away from the critical level, when $N^2 \gg \Omega_d^2$, is

$$\hat{w} = A_4(z-z_0)^{\frac{1}{2}+i\mu_m}\{1+a_{21}/(z-z_0)+a_{22}/(z-z_0)^2+\ldots\}$$
$$+B_4(z-z_0)^{\frac{1}{2}-i\mu_m}\{1+b_{21}/(z-z_0)+b_{22}/(z-z_0)^2+\ldots\},$$

where $\mu_m = (J_H+J_M-\frac{1}{4})^{\frac{1}{2}}$. This solution is the same as the one obtained by RV*c* in a perfectly conducting fluid. Hence the magnetic viscosity has no effect near and away from the critical level.

The total momentum flux is $M+G$, where

$$M = \rho_0\overline{uw}-\mu\overline{h_x h_z},\quad G = \rho_0\overline{vw}-\mu\overline{h_y h_z} \tag{5.4}$$

and an overbar denotes a time average. Expressing $M+G$ in terms of w, when dU/dz = constant, we get

$$M+G = \frac{\rho_0}{2}\mathrm{Re}\left\{\frac{(dU/dz)\,\nu_m \alpha^2 w w^*[-\nu_m^2\alpha^4\Omega_A^2 + \{k(k+l)/\alpha^2\}\,\Omega_A^2(\Omega_d^2 - \Omega_A^2 + \nu_m^2\alpha^4)]}{(\Omega_d^2+\nu_m^2\alpha^4)\,[\nu_m^2\alpha^2\,\Omega_d^2 + (\Omega_d^2 - \Omega_A^2)^2]}\right.$$
$$\left. - \frac{k+l}{\alpha^2}\frac{[\Omega_d\, dw/dz - (\mu H_0 l/\rho_0)\, dh_z/dz]\,(\Omega_d^2 - \Omega_A^2 + \nu_m^2\alpha^4)\,w^*}{[\nu_m\alpha^2\Omega_d + i(\Omega_d^2 - \Omega_A^2)]\,(\Omega_d + i\nu_m\alpha^2)}\right\}, \tag{5.5}$$

where $h_z = H_0 l w/(\Omega_d - i\nu_m\alpha^2)$ and w^* is the complex conjugate of w. By differentiating (5.5) with respect to z and using the wave equation (5.2) we find that

$$d(M+G)/dz \neq 0. \tag{5.6}$$

That is, the total momentum flux or the total stress is not constant. Hence the total momentum flux is not conserved. This is due to the dissipative effect of magnetic viscosity.

If U and N do not vary by very much over a wavelength, an internal gravity wave with horizontal wavenumbers k and l and vertical wavenumber m satisfies the dispersion relation

$$\Omega_d^3 - i\nu_m\alpha^2\,\Omega_d^2 - \frac{\alpha^2(N^2+\Omega_A^2)}{m^2+\alpha^2}\,\Omega_d + \frac{i\nu_m\alpha^4N^2}{m^2+\alpha^2} = 0.$$

When $\Omega_d \rightarrow 0$, $m \rightarrow \infty$ and the group velocity tends to zero, so that the waves are completely absorbed at $\Omega_d = 0$.

6. The time-dependent disturbance above a sinusoidal corrugation

In this section, we consider an initial-value problem to illustrate some of the results of the preceding sections. We consider a constant shear flow with the uniform applied magnetic field in the horizontal, y direction (figure 1). We consider a physical situation where the induced current sets up a polarized electric field which in turn balances the original induced field. In other words, we are considering a physical model similar to the open-circuit situation. This is an idealized physical model where the Lorentz forces have a negligible effect on the primary flow and a significant effect only on the disturbed flow. We take N^2 to be independent of z and the basic velocity $U(z)$ in the x direction to be of the form (see figure 1)

$$U = \begin{cases} U'h & \text{in region 1, i.e. } z > 2h, \\ U'(z-h) & \text{in region 2, i.e. } 0 < z < 2h. \end{cases} \tag{6.1}$$

The fluid is unbounded and there is only basic velocity U in the x direction, i.e.

$$w = 0 \quad \text{everywhere for} \quad t < 0. \tag{6.2}$$

At time $t = 0$ a disturbance is introduced by imposing a sinusoidal velocity distribution on the lower boundary at $z = 0$, and subsequently maintaining it:

$$w = a\cos kx \quad \text{on} \quad z = 0 \quad \text{for} \quad t > 0, \tag{6.3}$$

where a is the amplitude of the variation. The upper boundary condition is

$$w \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty \quad \text{for} \quad t > 0. \tag{6.4}$$

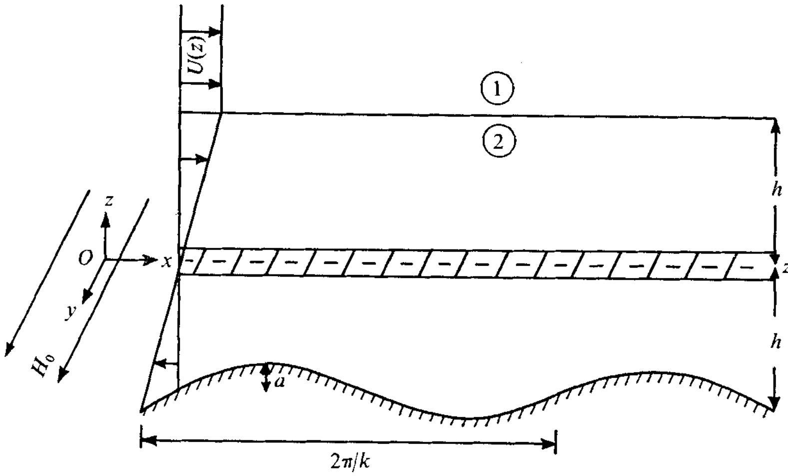

FIGURE 1. The basic state. ------, critical level; ﾃﾃﾃﾃ, critical layers.

A long time after the disturbance is introduced, it may be necessary to look at very large values of z to find a disturbance which is small, but we assume that, at any given time t, this is always possible. The source of the disturbance, as in the case of hydrodynamic flow (Booker & Bretherton 1967), is thus at $z = 0$.

The perturbation $w(x, y, z, t)$ satisfies the wave equation (2.3), which was obtained on the basis of a linearization valid only for small amplitudes a. For any given amplitude, however, it ultimately breaks down. Nevertheless, we investigate the solution up to the time that the theory becomes inconsistent, and may in principle justify this for any value of t by taking a sufficiently small. With the broken-line profile of (6.1), where we have assumed that the effect of Lorentz forces on the primary flow is negligible, U_{zz} is everywhere zero, except at the height $z = 2h$. At this level U_{zz} in the governing equation may be replaced by a delta function:

$$U_{zz} = -U'\delta(z-2h). \tag{6.5}$$

This is equivalent to matching the pressure and vertical velocity across the perturbed interface between the two separate fluids in regions 1 and 2.

We now introduce the dimensionless variables

$$\left.\begin{aligned} \xi = x/h, \quad \eta = y/h, \quad \zeta = (z-h)/h, \quad \tau = U't, \quad \alpha_0 = kh, \\ \beta_0 = lh, \quad \gamma = c/U'h, \quad S = A/U'h, \quad J = (N'^2/U'^2)(1+\beta_0^2/\alpha_0^2), \end{aligned}\right\} \tag{6.6}$$

where α_0 and β_0 are the dimensionless wavenumbers in the horizontal x and y directions respectively, γ is the dimensionless phase velocity in the y direction, S is the Alfvén number and J is the modified Richardson number. Use of the sinusoidal variations in ξ and η and the Laplace transform in time yields a convenient solution of the wave equation (2.3). Let

$$\left.\begin{aligned} w(\xi,\eta,\zeta,\tau) &= \mathrm{Re}\,[\tilde{w}(\zeta,\tau)\exp{(i(\alpha_0\xi+\beta_0\eta))}], \\ \hat{w}(\xi,\gamma) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^\infty \tilde{w}(\zeta,\tau)\exp{(i\alpha_0\gamma\tau)}\,d\tau. \end{aligned}\right\} \tag{6.7}$$

For the convergence of this integral we assume that the relevant part of the complex plane corresponds to $\gamma_i > 0$. The Laplace transform of the governing equation assumes different forms in regions 1 and 2. In region 1 ($\zeta > 1$)

$$\frac{i}{R_m}\frac{d^4\hat{w}}{d\zeta^4}+\left\{\frac{\alpha_0^2(1-\gamma)^2-S^2\beta_0^2}{\alpha_0(1-\gamma)}-\frac{i}{R_m}\left[2(\alpha_0^2+\beta_0^2)-\frac{J}{(1-\gamma)^2}\right]\right\}\frac{d^2\hat{w}}{d\zeta^2}$$
$$+\left\{\left[\frac{J}{(1-\gamma)^2}-(\alpha_0^2+\beta_0^2)\right]\left[\alpha_0(1-\gamma)-\frac{i}{R_m}(\alpha_0^2+\beta_0^2)\right]+\frac{S^2\beta_0^2(\alpha_0^2+\beta_0^2)}{\alpha_0(1-\gamma)}\right\}\hat{w}=0. \quad (6.8)$$

In region 2 ($-1 < \zeta < 1$)

$$\frac{i}{R_m}\frac{d^4\hat{w}}{d\zeta^4}+\left\{\frac{\alpha_0^2(\zeta-\gamma)^2-S^2\beta_0^2}{\alpha_0(\zeta-\gamma)}-\frac{i}{R_m}\left[2(\alpha_0^2+\beta_0^2)-\frac{J}{(\zeta-\gamma)^2}\right]\right\}\frac{d^2\hat{w}}{d\zeta^2}$$
$$+\frac{2}{\alpha_0(\zeta-\gamma)^3}\left\{S^2\beta_0^2(\zeta-\gamma)-\frac{2i\alpha_0 J}{R_m}\right\}\frac{d\hat{w}}{d\zeta}+\left\{\left[\frac{J}{(\zeta-\gamma)^2}-(\alpha_0^2+\beta_0^2)\right]\left[\alpha_0(\zeta-\gamma)\right.\right.$$
$$\left.\left.-\frac{i}{R_m}(\alpha_0^2+\beta_0^2)\right]+\frac{S^2\beta_0^2(\alpha_0^2+\beta_0^2)}{\alpha_0(\zeta-\gamma)}-\frac{2}{\alpha_0(\zeta-\gamma)^4}\left[S^2\beta_0^2(\zeta-\gamma)-\frac{i3\alpha_0\zeta}{R_m}\right]\right\}\hat{w}=0, \quad (6.9)$$

where $R_m = U'h^2/\nu_m$.

Now we need suitable boundary conditions. The continuity of pressure across the interface between regions 1 and 2 gives

$$\hat{w}_{1\zeta\zeta\zeta}-\hat{w}_{2\zeta\zeta\zeta}-\left\{\frac{iR_m}{\alpha_0(1-\gamma)}[\alpha_0^2(1-\gamma)^2-\beta_0^2S^2]+\left[(\alpha_0^2+\beta_0^2)-\frac{J}{(1-\gamma)^2}\right]\right\}(w_{1\zeta}-w_{2\zeta})$$
$$-\left\{\frac{iR_m}{\alpha_0(1-\gamma)^2}[\alpha_0^2(1-\gamma)^2-\beta_0^2S^2]-\frac{2J}{(1-\gamma)^3}\right\}\hat{w}=0 \quad \text{at} \quad \zeta=1, \quad (6.10)$$

which is quite different from the condition obtained by RVc for the case of a perfectly conducting fluid. The continuity of vertical velocity yields

$$\hat{w}_1=\hat{w}_2 \quad \text{at} \quad \zeta=1. \quad (6.11)$$

The boundary conditions (6.3) and (6.4) take the form

$$\hat{w}(\zeta,\tau)=-\frac{a}{(2\pi)^{\frac{1}{2}}}\frac{1}{i\alpha_0\gamma} \quad \text{on} \quad \zeta=-1, \quad (6.12)$$

$$\hat{w}\to 0 \quad \text{as} \quad \zeta\to\infty. \quad (6.13)$$

The pole $\gamma = 0$ in the above equation is a consequence of the specific time dependence assumed at the lower boundary. If the forcing at $z = 0$ is removed after a finite time τ_0 then

$$\hat{w}=-\frac{a}{(2\pi)^{\frac{1}{2}}}\frac{1}{i\alpha_0\gamma}(1-\exp\{-i\alpha_0\gamma\tau_0\}),$$

which has no singularity in the complex-γ plane.

The remaining boundary conditions will depend on the perturbed magnetic field. The continuity of vertical magnetic field h_z across the interface between regions 1 and 2 gives

$$\hat{w}_{1\zeta\zeta}=\hat{w}_{2\zeta\zeta} \quad \text{at} \quad \zeta=1. \quad (6.14)$$

Further, since h_x, h_y and hence dh_z/d_z are continuous, we have

$$\hat{w}_{1\zeta\zeta\zeta}-\hat{w}_{2\zeta\zeta\zeta}-\{\alpha_0^2+\beta_0^2-J/(1-\gamma)^2-i\beta_0^2S^2R_m/\alpha(1-\gamma)\}(\hat{w}_{1\zeta}-\hat{w}_{2\zeta}) + \{2J/(1-\gamma)^3+i\beta_0^2S^2R_m/\alpha(1-\gamma)^2\}\hat{w}=0 \quad \text{at} \quad \zeta=1. \tag{6.15}$$

From the magnetic induction equation, using the continuity of d^2hz/dz^2, we get

$$\hat{w}_{1\zeta\zeta\zeta\zeta}-\hat{w}_{2\zeta\zeta\zeta\zeta}+\frac{2i}{\alpha_0(1-r)^3}\{R_mS^2\beta_0^2(1-r)-2i\alpha_0J\}\hat{w}_{2\zeta} - \frac{2i}{\alpha_0(1-\gamma)^4}\{S^2\beta_0^2R_m(1-\gamma)-i3\alpha_0J\}\hat{w}=0. \tag{6.16}$$

In region 1, the solution of (6.8) consistent with the upper boundary condition is

$$\hat{w}=A_1\exp[im_1(\zeta-1)]+B_1\exp[im_2(\zeta-1)], \tag{6.17}$$

$$m_{1,2}=\left(\frac{R_m}{2}\right)^{\frac{1}{2}}\left\{\frac{1}{R_m}\left[\frac{J}{(1-\gamma)^2}-2(\alpha_0^2+\beta_0^2)\right]-\frac{i[\alpha_0^2(1-\gamma)^2-S^2\beta_0^2]}{\alpha_0(1-\gamma)} \pm\left[4S^2\beta_0^2+\left(\frac{J}{R_m(1-\gamma)^2}+\frac{i[\alpha_0^2(1-\gamma)^2+S^2\beta_0^2]}{\alpha_0(1-\gamma)}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

with $m_{1i}>0$ and $m_{2i}>0$ when $\gamma_i>0$. This branch is forced by the vanishing of $\hat{w}$ for large ζ under the condition for which the Laplace transform (6.7) is convergent, i.e. $\gamma_i>0$.

In region 2, we note that $\zeta=\gamma$ is the regular singular point of (6.9). Hence the solution of (6.9) near $\zeta=\gamma$ can be obtained using the Frobenius method and is

$$\hat{w}=A_2(\zeta-\gamma)^{\frac{1}{2}+i\mu_0}I_1(\zeta-\gamma)+B_2(\zeta-\gamma)^{\frac{1}{2}-i\mu_0}I_2(\zeta-\gamma) + C_2(\zeta-\gamma)^3I_3(\zeta-\gamma)+D_2(\zeta-\gamma)^2I_4(\zeta-\gamma), \tag{6.18}$$

where the functions I_i ($i=1,2,3,4$) are given by

$$\begin{bmatrix} I_1\\ I_2\\ I_3\\ I_4\end{bmatrix}=\begin{bmatrix}1 & a_{21} & a_{22} & \cdots\\ 1 & b_{21} & b_{22} & \cdots\\ 1 & c_{21} & c_{22} & \cdots\\ 1 & d_{21} & d_{22} & \cdots\end{bmatrix}\begin{bmatrix}1\\ \zeta-\gamma\\ (\zeta-\gamma)^2\\ \vdots\end{bmatrix},$$

where $a_{21}, a_{22}, \ldots, b_{21}, b_{22}, \ldots,$ and so on are all known constants. The constants A_1, B_1, A_2, B_2, C_2 and D_2 can be determined using the above boundary conditions and they are not presented here since the expressions are very lengthy.

The complete formal solution to the problem is then given by the inverse Laplace transform

$$w(\xi,\eta,\zeta,\tau)=\mathrm{Re}\left\{\frac{1}{(2\pi)^{\frac{1}{2}}}\exp[i(\alpha_0\xi+\beta_0\eta)]\int_\Gamma\hat{w}(\gamma,\zeta)\exp(-i\alpha_0\gamma\tau)\,d\gamma\right\}, \tag{6.19}$$

where the contour of integration Γ lies along the real-γ axis from $-\infty$ to ∞, except where there is a singularity in the integrand, in which case it lies above (i.e. $\gamma_i>0$).

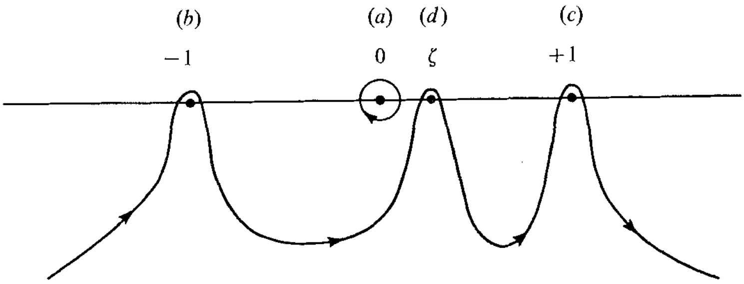

FIGURE 2. The deformed contour of integration.

7. Wave propagation after a long time

As in the case of a perfectly conducting fluid discussed by RVc, the integral (6.19) is in general mathematically complicated, but if τ is large methods of asymptotic analysis akin to that of steepest descent (Jeffreys & Jeffreys 1946, § 17.04) may be applied to give great simplifications. The dominant contributions to the integral come from neighbourhoods in the complex-γ plane of points where either the integral is singular or the derivative with respect to γ of the coefficient of τ in the exponent in (6.19) vanishes (saddle point). If ζ is kept finite as $\tau \to \infty$ there are no saddle points and the largest contribution comes from the pole at $\gamma = 0$. This situation is the same as the one discussed by Booker & Bretherton (1967) in the case of hydrodynamic flow and the one discussed by RVc in the case of perfectly conducting flow in the presence of an aligned magnetic field. In this section, we try to show that, except in a neighbourhood of the critical level at $\zeta = \gamma$ (i.e. $\zeta = 0$) which shrinks with time, the motion everywhere becomes that of a standing wave pattern, there being several small decaying wave motions superposed on this steady wave motion. If the Richardson number $J_H \geqslant 1$, the waves above the layer are very much reduced in magnitude. We note that one of the decaying oscillations is the remnant of transient waves induced by the impulsive start to the motions, which are absorbed in the shear layer, each at the critical level appropriate to their frequency.

There is a region above and below the critical level $\zeta = \gamma$ which decreases in thickness as time goes on in which the motion is not yet steady, even to a first approximation. We shall call this region the critical layer. Above and below it the motion is damped because of magnetic viscosity and hence a steady state is achieved quite quickly. Although the flux of total momentum is not conserved because of magnetic viscosity, the total momentum incident on the critical layer associated with the upward-travelling wave is nearly all transferred into the mean flow and the wave is effectively absorbed. This is contrary to the perfectly conducting case (RVc), where the waves are not absorbed in the critical layer $\zeta = \gamma$. We can also see that in the critical layer the maximum magnitude of the horizontal velocities increases with time, this increase in velocity being small compared with the hydrodynamic velocity, but after any finite interval the velocities and their spatial derivatives are everywhere finite and well behaved. Further, if ζ/τ is kept constant as $\tau \to \infty$, the largest contribution comes from a saddle point. This corresponds to an upward-travelling dispersive group of waves whose dominant frequency at any point is exactly such that the corresponding vertical component of group velocity is ζ/τ. Above this group the disturbance has not yet penetrated and below it the steady-state solution is achieved. It describes the influence of the impulsive start to the solution, but ultimately passes by any given point.

The above results can be proved by considering the singularities of the integrand in (6.19). These are:

(*a*) $\gamma = 0$, a pole, arising from the applied boundary condition (6.3);
(*b*) $\gamma = -1$, a branch point;
(*c*) $\gamma = +1$, a branch point plus an essential singularity, $l_1, l_2 \to \infty$;
(*d*) $\gamma = \zeta$, a branch point.

In addition there are possibly poles when $l_1, l_2 = 0$. Assuming that singularity (*d*) does not coincide with any of the others, we deform the contour of integration Γ according to figure 2 so that all the singularities lie in the region $\gamma_i < 0$. As $\tau \to \infty$ the integrand is exponentially small except in those regions which are near the real axis $\gamma_i = 0$.

For fixed ζ, the largest contribution to the integral in (6.19) comes from the pole at $\gamma = 0$. Thus as $\tau \to \infty$, equation (6.19) takes the following forms.

In region 1 we obtain

$$w \sim \operatorname{Re}\{(2\pi)^{\frac{1}{2}} i[A_1'(0) \exp(im_{10}(\zeta - 1)) + B_1'(0) \exp(im_{20}(\zeta - 1))] \exp[i(\alpha_0 \xi + \beta_0 \eta)]\}, \tag{7.1}$$

where m_{10} and m_{20} are obtained from (6.17) by setting $\gamma = 0$. This represents two stationary waves above the critical level. In region 2

$$w \sim \operatorname{Re}\{(2\pi)^{\frac{1}{2}} i[A_2'(0)\,\zeta^{\frac{1}{2}+i\mu_0} I_1(\zeta) + B_2'(0)\,\zeta^{\frac{1}{2}-i\mu_0} + C_2'(0)\,\zeta^3 I_3(\zeta) + D_2'(0)\,\zeta^2 I_4(\zeta)] \exp[i(\alpha_0 \xi + \beta_0 \eta)]\}, \tag{7.2}$$

where $A_2'(0) = \lim_{\gamma \to 0} (\gamma A_2(\gamma))$. The first and the second terms represent standing waves above and below the critical level respectively and the amplitude of these waves is reduced by a factor $\exp(-\mu_0 \pi)$ in passing through the critical level. The other two terms tend to zero near the critical level $\zeta = \gamma$ (i.e. $\zeta = 0$).

The contributions to the integral in (6.19) associated with the remaining singularities all tend to zero as $\tau \to \infty$ for fixed ζ. Following the analysis of RV*c*, however, we can obtain the contribution from the singularity (*d*) (i.e. $\zeta = \gamma$), which is given by

$$w(\xi, \eta, \zeta, \tau) = \operatorname{Re}\left\{\frac{\exp[i(\alpha_0 \xi + \beta_0 \eta - \alpha_0 \zeta \tau)]}{(2\pi)^{\frac{1}{2}}} \left[\frac{3!\,A_1}{-\alpha_0^3} \frac{1}{\tau^4} + \frac{2!\,iB_1}{-\alpha_0^2} \frac{1}{\tau^3} + \frac{(\frac{1}{2} + i\mu_0)!\,iC_1}{(i\alpha)^{\frac{1}{2}+i\mu_0}} \frac{1}{\tau^{\frac{3}{2}+i\mu_0}} + \frac{(\frac{1}{2} + i\mu_0)!}{(i\alpha)^{\frac{1}{2}-i\mu_0}} \frac{1}{\tau^{\frac{3}{2}-i\mu_0}}\right]\right\}. \tag{7.3}$$

This describes a locally plane sinusoidal wave with phase $\alpha_0 \xi + \beta_0 \eta - \alpha_0 \zeta \tau$. The lines of constant phase are advected with the basic flow velocity and tilted to a nearly horizontal orientation as $\tau \to \infty$. The horizontal wavenumbers remain constant, but the vertical wavenumber becomes large and the vertical velocity decays with time. Similarly, we can show that the contributions from the remaining singularities all tend to zero as $\tau \to \infty$.

The above analysis is confined to the case where the singularities are distinct. Even when the singularities coincide with one another (6.19) can be integrated (see RVc) and the corresponding wave decays to zero as $\tau \to \infty$.

8. Transient disturbances in a conducting shear flow

Although an analysis similar to that of §§ 6 and 7 may be used to describe the disturbance due to a transient stimulus in a conducting shear layer, the resulting integral, which is the formal solution, is mathematically complicated and cannot be evaluated in terms of elementary functions. Unfortunately, the addition of a finite electrical conductivity makes the problem even more hopelessly intractable. However, it is possible to make, as in the case of a perfectly conducting fluid (RVc), some general statements about the velocity distribution.

The asymptotic solution for the velocity distribution discussed in §§ 6 and 7 has a pole at $\gamma = 0$ because of the lower boundary condition $w = a \cos kx$. However, in the case of transient disturbances this pole disappears, so that the velocities everywhere decay with time and the dominant contributions come from the singularities of the type (b) and (c), which decay with time. Even these singularities will not be present in an unbounded uniform shear flow, where they are associated with decaying oscillations which are coherent over the whole fluid.

In the present case the vertical velocity associated with the singularity (d) can be shown to be of the form

$$w = \operatorname{Re}\left[\frac{F_1}{t^4} + \frac{F_2}{t^3} + \frac{F_3}{t^{\frac{3}{2}+i\mu_0}} + \frac{F_4}{t^{\frac{3}{2}-i\mu_0}}\right] \exp\left[i(kx + ly - kUt)\right]. \tag{8.1}$$

Each term in (8.1) describes locally plane waves of very small vertical wavelength ($= -1/2\pi ktU_z$) which decay to zero as $t \to \infty$. This decay of the vertical velocity field is due to the manifestation of critical-layer absorption for a continuous spectrum of frequencies. Each frequency is associated with a critical level z_0 and at each height z there is a corresponding frequency ω for which it is critical.

9. Conclusions

It has been shown that the governing wave equation is one of order four and is singular only at $\Omega_d = 0$ but not at $\Omega_d = \pm\Omega_A$; whereas in the case of perfectly conducting fluid discussed in RVc the wave equation is of order two and is singular at $\Omega_d = 0, \pm\Omega_A$. From this, we conclude that the effect of magnetic viscosity on the flow is to remove only the magnetic singularities $\Omega_d = \pm\Omega_A$ and not the hydrodynamic singularity $\Omega_d = 0$. We find that the behaviour of waves near the hydrodynamic critical level $\Omega_d = 0$ is the same as in the case of hydrodynamic inviscid shear flow discussed by Booker & Bretherton (1967); whereas the behaviour of waves away from the critical levels $\Omega_d = 0, \pm\Omega_A$ is similar to the perfectly conducting fluid case discussed in RVc. From this we conclude that the effect of electrical conductivity on waves is negligible near the critical level $\Omega_d = 0$ and away from $\Omega_d = 0, \pm\Omega_A$. Although the wave equation is not singular at $\Omega_d = \pm\Omega_A$, we find that these points are still significant in connexion with the asymptotic solutions of the wave equation, because they are the magnetic Stokes points for such solutions. The limiting form of the wave equation near the Stokes points and an exact solution of this equation has been derived and will be presented in the subsequent paper (Rudraiah & Venkatachalappa 1974).

The mechanism of wave absorption near the critical level has been studied through the group-velocity approach. We have found that the vertical wave-number becomes infinite as the wave approaches the critical level $\Omega_d = 0$ and the group velocity becomes zero at that point, so that the waves are completely absorbed there; whereas the waves are transmitted across the magnetic Stokes points $\Omega_d = \pm\Omega_A$. This is contrary to the perfectly conducting fluid case (RVc), where the waves are completely absorbed at $\Omega_d = \pm\Omega_A$ and there exists a forbidden zone, namely $|\Omega_d| < \Omega_A$, for the propagation of waves.

The energy flux $\overline{Pw}$ becomes infinite at $\Omega_d = 0$, which is analogous to the situation discussed by Miles (1961). This may be due to the neglect of viscosity and heat conduction and the energy flux may become finite in the presence of these two diffusive effects. In particular we note that the discussion of momentum transport to the basic flow becomes very difficult since the expression for the vertical momentum flux is mathematically complicated. Therefore, to study the momentum transport analytically we have used the approximation

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2} \gg \frac{\partial^2 h}{\partial z^2},$$

where h is the perturbation magnetic field. Within this approximation we have found that the vertical momentum flux is not conserved and this may be due to the dissipative effect of magnetic diffusivity. Even in this approximation the group-velocity approach shows that the waves are absorbed at the critical level $\Omega_d = 0$.

In §§ 6 and 7 we have considered an initial-value problem and found that near the critical level $\zeta = \gamma$ the motion becomes that of a standing wave pattern.

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